

HOLOMORPHIC SUBMERSIONS FROM STEIN MANIFOLDS

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1. Introduction.

Recall that a *Stein manifold* is a complex manifold biholomorphic to a closed complex submanifold of a complex Euclidean space \mathbb{C}^N [GR, p. 226]. A holomorphic map $f: X \rightarrow Y$ whose differential $df_x: T_x X \rightarrow T_{f(x)} Y$ is surjective for every $x \in X$ is said to be a *submersion* of X to Y . The following was proved in [F1, Theorem II]; for $n = q = 1$ see also [GN]:

A Stein manifold X admits a holomorphic submersion to \mathbb{C}^q for some $q < \dim X$ if and only if its tangent bundle TX admits a surjective complex vector bundle map onto the trivial rank q bundle $\tau_X^q = X \times \mathbb{C}^q$.

The necessity of the latter condition is clear since the tangent map of a submersion $X \rightarrow \mathbb{C}^q$ induces a surjective vector bundle map $TX \rightarrow \tau_X^q$. The corresponding problem for $n = q > 1$ remains open [F1].

In this paper we consider the analogous problem with the target space \mathbb{C}^q replaced by a more general complex manifold Y . The following result in the smooth category was proved by A. Phillips [P] and M. Gromov [Gr1, Gr3]:

A continuous map $f_0: X \rightarrow Y$ from a smooth open manifold X to a smooth manifold Y is homotopic to a submersion $f_1: X \rightarrow Y$ if and only if there exists a surjective vector bundle map $\iota_0: TX \rightarrow f_0^ TY$ over X , i.e., the pull-back $f_0^* TY$ is a quotient bundle of TX . The regular homotopy classes of submersions $X \rightarrow Y$ are classified by the homotopy classes of their tangent maps.*

In the holomorphic category the topological condition on the existence of ι_0 does not suffice. For example, a Kobayashi-hyperbolic manifold Y admits no nonconstant holomorphic images of \mathbb{C}^n but the topological condition trivially holds for the constant map $\mathbb{C}^n \rightarrow y_0 \in Y$. This suggests that we restrict the attention to manifolds which admit ‘sufficiently many’ holomorphic submersions $\mathbb{C}^n \rightarrow Y$ for a given $n \geq \dim Y$. It turns out that a suitably precise form of this condition suffices for a complete analogue of the Gromov-Phillips theorem, thus justifying the following heuristic principle (a form of the Oka principle):

If a complex manifold Y admits sufficiently many holomorphic submersions $\mathbb{C}^n \rightarrow Y$ for a given integer $n \geq \dim Y$ then the existence of holomorphic submersions $X \rightarrow Y$ from n -dimensional Stein manifolds X reduces to a purely topological problem.

We say that a complex manifold Y satisfies *Property S_n* for some integer $n \geq \dim Y$ if any holomorphic submersion $O \rightarrow Y$ from a compact convex set $O \subset \mathbb{C}^n$ of a certain special type can be approximated uniformly on O by holomorphic submersions $\mathbb{C}^n \rightarrow Y$ (Definition 1 in Sect. 2). The following is a special case of our main result, Theorem 2.1:

THEOREM. *Assume that X is an n -dimensional Stein manifold and Y is a complex manifold satisfying Property S_n . A continuous map $f: X \rightarrow Y$ is homotopic to a holomorphic submersion of X to Y if and only if there exists a surjective complex vector bundle map $\iota: TX \rightarrow f^*TY$.*

Furthermore, *Property S_n is both necessary and sufficient for a stronger version of the above theorem with approximation on compact $\mathcal{O}(X)$ -convex subsets of the source manifold X* ; Theorem 2.1 (b). If Y also satisfies the analogous *Property HS_n* concerning the approximation of homotopies of holomorphic submersions $O \rightarrow Y$ by homotopies of submersions $\mathbb{C}^n \rightarrow Y$ then for any n -dimensional Stein manifold X the regular homotopy classes of holomorphic submersions $X \rightarrow Y$ are classified by the homotopy classes of their tangent maps (Corollary 2.4).

If $n = \dim X \geq 2 \dim Y - 1$ then f^*TY is always a quotient of TX by topological reasons and hence every continuous map $X \rightarrow Y$ is homotopic to a holomorphic submersion (Corollary 2.3).

One cannot use Property S_n directly since a general submersion $X \rightarrow Y$ does not factor as $X \rightarrow \mathbb{C}^n \rightarrow Y$. Instead such decompositions are used on small subsets of X and the resulting local submersions of X to Y are pieced together into a global submersion by the analytic tools developed in [F1] and in this paper.

Properties S_n and HS_n are equivalent to apparently weaker conditions on uniform approximability of submersions $O \rightarrow Y$ on special compact convex sets $O \subset \mathbb{C}^n$ by submersions $\mathbb{C}^n \setminus A \rightarrow Y$, where $A \subset \mathbb{C}^n$ is an algebraic subvariety of codimension at least two which does not intersect O (this relies upon the theory of holomorphic automorphisms of \mathbb{C}^n). In Sect. 5 we establish Properties S_n and HS_n when $n > \dim Y = q$ (or $n = q = 1$) and Y is any of the following: \mathbb{C}^q , \mathbb{CP}^q , a complex Grassmannian, a Zariski open set with thin complement (containing no hypersurfaces) in any of the above, or a holomorphic quotient of any of the above (this class contains all complex tori and Hopf manifolds).

For Riemann surfaces we obtain a complete answer by proving that the following are equivalent (Corollary 2.8):

- (a) Y is one of the Riemann surfaces \mathbb{CP}^1 , \mathbb{C} , $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, or a complex torus (the quotient \mathbb{C}/Γ by a rank two lattice $\Gamma \subset \mathbb{C}$);

(b) any continuous map from a Stein manifold X to Y is homotopic to a holomorphic submersion.

The Riemann surfaces listed in (a) are precisely those which are not Kobayashi hyperbolic. When Y is \mathbb{C}^* or a torus our result is new even when X is an open Riemann surface.

The heuristic principle behind our main result is reminiscent of Gromov's extension of the Oka-Grauert principle [Gr4]: *The existence of many dominating holomorphic maps $\mathbb{C}^n \rightarrow Y$ implies the existence of many holomorphic maps $X \rightarrow Y$ from any Stein manifold X .* A dominating map $s: \mathbb{C}^n \rightarrow Y$ based at a point $y \in Y$ is one for which $s(0) = y$ and $ds_0: T_0 \mathbb{C}^n \rightarrow T_y Y$ is surjective (i.e., s is a submersion near $0 \in \mathbb{C}^n$). A *dominating spray* on Y is a family of such maps depending holomorphically on $y \in Y$ (its domain is a holomorphic vector bundle over Y). Comparing with Property S_n we see that the two conditions have similar flavour. For the Oka-Grauert-Gromov theory we refer to [G3, Gr4, HL2, FP1, FP2, FP3, F3, W].

An important ingredient in our construction of submersions is a holomorphic approximation theorem on certain handlebodies in arbitrary complex manifolds; Theorem 3.2 in Sect. 3. This result, together with a geometric lemma from [F1], gives an approximate extension of the submersion across a critical level of a strongly plurisubharmonic exhaustion function in the source manifold. In Sect. 6 we use the same method to give a simple proof of the Oka-Grauert principle for sections of holomorphic fiber bundles over Stein manifolds whose fiber admits a finite dominating family of sprays.

2. The main results.

We denote by $\mathcal{O}(X)$ the algebra of all holomorphic functions on a complex manifold X . A function (or map) is holomorphic on a closed subset K in X if it is holomorphic in some open neighborhood of K ; the set of all such functions (with the usual identification of functions which agree in a neighborhood of K) will be denoted $\mathcal{O}(K)$. Any statement concerning a holomorphic map on a closed set should be understood in the sense that it holds in some open neighborhood; for homotopies of maps the neighborhood is the same for all maps in the homotopy. A compact set $K \subset X$ is $\mathcal{O}(X)$ -convex if for every $p \in X \setminus K$ there exists $f \in \mathcal{O}(X)$ such that $|f(p)| > \sup_K |f|$.

A *homotopy of holomorphic submersions* $X \rightarrow Y$ is a family of holomorphic submersions $f_t: X \rightarrow Y$ ($t \in [0, 1]$) depending continuously on t . It follows that the tangent maps $Tf_t: TX \rightarrow TY$ are also continuous in t .

Let $z = (z_1, \dots, z_n)$ be the coordinates on \mathbb{C}^n , with $z_j = x_j + iy_j$. Set

$$Q = \{z \in \mathbb{C}^n: |x_j| \leq 1, |y_j| \leq 1, j = 1, \dots, n\}. \quad (2.1)$$

A *special convex set* in \mathbb{C}^n is a compact convex subset of the form

$$O = \{z \in Q: y_n \leq h(z_1, \dots, z_{n-1}, x_n)\}, \quad (2.2)$$

where h is a smooth (weakly) concave function with values in $(-1, 1)$.

DEFINITION 1. Let d be a distance function induced by a smooth Riemannian metric on complex manifold Y .

- (a) Y satisfies Property S_n if for any holomorphic submersion $f: O \rightarrow Y$ on a special convex set $O \subset \mathbb{C}^n$ and any $\epsilon > 0$ there is a holomorphic submersion $\tilde{f}: Q \rightarrow Y$ satisfying $\sup_{x \in O} d(f(x), \tilde{f}(x)) < \epsilon$.
- (b) Y satisfies Property HS_n if for any homotopy of holomorphic submersions $f_t: O \rightarrow Y$ ($t \in [0, 1]$) such that f_0 and f_1 extend to holomorphic submersions $Q \rightarrow Y$ there exists for any $\epsilon > 0$ a homotopy of holomorphic submersions $\tilde{f}_t: Q \rightarrow Y$ ($t \in [0, 1]$) satisfying $\tilde{f}_0 = f_0$, $\tilde{f}_1 = f_1$, and $\sup_{x \in O, t \in [0, 1]} d(f_t(x), \tilde{f}_t(x)) < \epsilon$.

An obvious induction shows that Property S_n of Y implies the following: *Any holomorphic submersion $f: O \rightarrow Y$ on a special convex set $O \subset \mathbb{C}^n$ can be approximated uniformly on O by holomorphic submersions $\mathbb{C}^n \rightarrow Y$.* The analogous remark holds for Property HS_n .

Let X and Y be complex manifolds. Denote by $\mathcal{S}(X, Y)$ the set of all pairs (f, ι) where $f: X \rightarrow Y$ is a continuous map and $\iota: TX \rightarrow TY$ is a fiberwise surjective complex vector bundle map making the following diagram commute:

$$\begin{array}{ccc} TX & \xrightarrow{\iota} & TY \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Figure 1: The set $\mathcal{S}(X, Y)$

Such ι is the composition of a surjective complex vector bundle map $TX \rightarrow f^*TY$ with the natural map $f^*TY \rightarrow TY$. Let $\mathcal{S}_{holo}(X, Y)$ consist of all pairs $(f, Tf) \in \mathcal{S}(X, Y)$ where $f: X \rightarrow Y$ is a holomorphic submersion and Tf its tangent map. We equip $\mathcal{S}(X, Y)$ with the compact-open topology.

The following is our main theorem.

THEOREM 2.1. Assume that X is a Stein manifold of dimension n and Y is a complex manifold satisfying Property S_n .

- (a) (Existence of submersions) Every $(f_0, \iota_0) \in \mathcal{S}(X, Y)$ can be connected by a path $\{(f_t, \iota_t)\}_{t \in [0, 1]} \subset \mathcal{S}(X, Y)$ to some $(f_1, Tf_1) \in \mathcal{S}_{holo}(X, Y)$.
- (b) (Approximation) If $K \subset X$ is a compact $\mathcal{O}(X)$ -convex subset, $f_0|_K$ is a holomorphic submersion, and $\iota_0|_K = Tf_0|_K$ then the path (f_t, ι_t) in (a) can be chosen such that for every $t \in [0, 1]$, $f_t|_K$ is a holomorphic submersion uniformly close to $f_0|_K$ and $\iota_t|_K = Tf_t|_K$.
- (c) (Regular homotopies of submersions) If Y satisfies Property HS_n then any path $\xi_t = (f_t, \iota_t) \in \mathcal{S}(X, Y)$ ($t \in [0, 1]$) with $\xi_0, \xi_1 \in \mathcal{S}_{holo}(X, Y)$ can be deformed with fixed ends to a path in $\mathcal{S}_{holo}(X, Y)$.

Theorem 2.1 is proved in Section 4. In the special case $Y = \mathbb{C}^q$, with $n > q$ or $n = q = 1$, parts (a) and (b) were proved in [F1] (for $n = q = 1$ see also

[GN]), but part (c) is new even in this case. The relevant Property HS_n holds for \mathbb{C}^q whenever $n > q$ (Proposition 2.5 below).

Remarks. 1. If the conclusion of part (b) in Theorem 2.1 holds for a given complex manifold Y , with K any compact convex set in $X = \mathbb{C}^n$, then (by definition) Y satisfies Property S_n . Hence both the topological condition (the existence of ι) and the analytic condition (Property S_n of Y) are necessary for parts (a) and (b) in Theorem 2.1.

2. Clearly $HS_n \Rightarrow S_n$. We don't know whether the converse always holds, but in all examples for which we prove S_n we also prove HS_n .

Part (a) of Theorem 2.1 implies the following.

COROLLARY 2.2. *If X is an n -dimensional Stein manifold and Y satisfies Property S_n then for any $(f, \iota) \in \mathcal{S}(X, Y)$ there is a nonsingular holomorphic foliation of X whose normal bundle is isomorphic to f^*TY (as a topological complex vector bundle over X).*

Such a foliation is given by the level sets of a submersion $X \rightarrow Y$ furnished by Theorem 2.1 (a). The corresponding result for foliations with trivial normal bundle (and $Y = \mathbb{C}^q$) was obtained in [F1].

If $\dim X \geq 2\dim Y - 1$ then every map $f: X \rightarrow Y$ can be covered by a map $\iota: TX \rightarrow TY$ such that $(f, \iota) \in \mathcal{S}(X, Y)$. This follows by standard topological methods from the fact that an n -dimensional Stein manifold is homotopic to an n -dimensional CW-complex; it will also be clear from our proof of Theorem 2.1. Hence Theorem 2.1 implies

COROLLARY 2.3. *If Y satisfies Property S_n for some $n \geq 2\dim Y - 1$ then any map $f: X \rightarrow Y$ from an n -dimensional Stein manifold X is homotopic to a holomorphic submersion of X to Y .*

Combining parts (a) and (c) in Theorem 2.1 we also obtain

COROLLARY 2.4. *If a complex manifold Y satisfies Property HS_n then for every n -dimensional Stein manifold X the natural inclusion*

$$\mathcal{S}_{holo}(X, Y) \hookrightarrow \mathcal{S}(X, Y)$$

induces a bijective correspondence of the path-connected components of the two spaces.

Comparing with the Oka-Grauert principle [G3, Gr4, FP1] one might expect that the above inclusion is a *weak homotopy equivalence*; however, our proof does not give this much.

In the remainder of this section we discuss the question which manifolds satisfy Properties S_n and HS_n . For Riemann surfaces both properties hold precisely on the non-hyperbolic ones (Corollary 2.8). For manifolds of dimension > 1 a complete answer seems out of reach even in the class of projective algebraic surfaces. The following result gives some nontrivial examples.

PROPOSITION 2.5. *The following manifolds Y^q satisfy Properties S_n and HS_n for any $n > q$, as well as for $n = q = 1$:*

- (a) \mathbb{C}^q , \mathbb{CP}^q , or a complex Grassmannian;
- (b) a Zariski open subset in any of the manifolds from (a) whose complement contains no complex hypersurfaces.

For a proof see Sect. 5. The following simple observation will be useful.

PROPOSITION 2.6. *Let $\pi: \tilde{Y} \rightarrow Y$ be a holomorphic covering. If one of the manifolds Y , \tilde{Y} satisfies Property S_n (resp. HS_n) then so does the other.*

Proof. Assume first that \tilde{Y} satisfies S_n . Given a holomorphic submersion $f: O \rightarrow Y$ from a special compact convex set $O \subset \mathbb{C}^n$, there is a holomorphic lifting $g: O \rightarrow \tilde{Y}$ (satisfying $\pi \circ g = f$) which is also a submersion since π is locally biholomorphic. By the assumed Property S_n of \tilde{Y} we can approximate g by an entire submersion $\tilde{g}: \mathbb{C}^n \rightarrow \tilde{Y}$, and then $\tilde{f} = \pi \circ \tilde{g}: \mathbb{C}^n \rightarrow Y$ is a submersion such that $\tilde{f}|_O$ approximates f . This proves that Y also satisfies S_n .

Conversely, assume that Y satisfies S_n . Given a submersion $g: O \rightarrow \tilde{Y}$, we approximate the submersion $f = \pi \circ g: O \rightarrow Y$ by a submersion $\tilde{f}: \mathbb{C}^n \rightarrow Y$ (using Property S_n of Y) and then lift \tilde{f} to a (unique!) map $\tilde{g}: \mathbb{C}^n \rightarrow \tilde{Y}$ satisfying $\tilde{g}(z_0) = g(z_0)$ for some point $z_0 \in O$. Then \tilde{g} is a holomorphic submersion which approximates g on O .

The same arguments hold for homotopies of submersions which gives the corresponding statement for Property HS_n . This proves Proposition 2.6.

Combining Propositions 2.5 and 2.6 we obtain

COROLLARY 2.7. *A complex manifold Y whose universal covering space is biholomorphic to \mathbb{C}^q , or to $\mathbb{C}^q \setminus A$ for an algebraic subvariety of codimension at least two, satisfies Properties S_n and HS_n when $n > q$ or $n \geq q = 1$.*

The manifolds covered by \mathbb{C}^q are of the form $\mathbb{C}^k \times (\mathbb{C}^*)^l \times T^s$ ($k + l + s = q$), where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and T^s is a complex torus, i.e., $T^s = \mathbb{C}^s / \Gamma$ for a lattice $\Gamma \subset \mathbb{C}^s$ of rank $2s$. Recall that every *Hopf manifold* is a holomorphic quotient of $\mathbb{C}^q \setminus \{0\}$ and hence Theorem 2.1 applies (see [B, p. 172]).

The following result, which follows from Proposition 2.5 and Corollary 2.7, is a complete solution to the submersion problem if Y is a Riemann surface.

COROLLARY 2.8. *If Y is any of the Riemann surfaces \mathbb{CP}^1 , \mathbb{C} , \mathbb{C}^* , or a complex torus \mathbb{C}/Γ then any continuous map $f_0: X \rightarrow Y$ from a Stein manifold X is homotopic to a holomorphic submersion $f_1: X \rightarrow Y$. Furthermore, if $f_0|_K: K \rightarrow Y$ is a holomorphic submersion for a compact $\mathcal{O}(X)$ -convex subset $K \subset X$ then the homotopy can be chosen to approximate f_0 uniformly on K . Conversely, if a Riemann surface Y admits a nonconstant holomorphic map $\mathbb{C} \rightarrow Y$ then Y belongs to the above list.*

Indeed, the universal covering of any Riemann surface is either \mathbb{CP}^1 , \mathbb{C} or the disc, and only the first two admit a nonconstant holomorphic image of \mathbb{C} ; their holomorphic quotients are listed in Corollary 2.8.

The manifolds for which we prove Property S_n in this paper are all *subelliptic* in the sense of [F3], i.e., they admit a finite dominating family of holomorphic sprays; hence by the main result in [F3] any continuous map $X \rightarrow Y$ is homotopic to a holomorphic map (the Oka-Grauert principle).

Problem 1. Does every subelliptic manifold Y satisfy Property S_n for all $n > \dim Y$?

A good test case may be complex Lie groups G ; a dominating holomorphic spray on G is given by $s(g, v) = \exp(v)g$ where $g \in G$ and v is a vector in the Lie algebra of G .

Problem 2. Does a manifold Y which satisfies S_n for some $n \geq \dim Y$ also satisfy S_k for $k > n$?

Problem 3. Is the Property S_n invariant with respect to proper modifications such as blow-ups and blow-downs?

3. Holomorphic approximation on handlebodies.

The main result of this section, Theorem 3.2, concerns holomorphic approximation of mappings between complex manifolds or, more generally, of sections of holomorphic submersions, on certain compact sets obtained by attaching a totally real submanifold to a compact holomorphically convex subset. Theorem 3.2, which generalizes a result of Hörmander and Wermer [HW, Theorem 4.1], will be used in the proof of Theorem 2.1 in Sect. 4. The proof uses Hörmander's L^2 -solution to the $\bar{\partial}$ -equation [Hö1, Hö2] on special Stein neighborhoods of S furnished by Theorem 3.1 below. For $X = \mathbb{C}^n$ Theorem 3.1 is due to Hörmander and Wermer [HW, Theorem 3.1]; here we extend it to arbitrary complex manifolds.

A compact set K in a complex manifold X is said to be *holomorphically convex* if K has an open Stein neighborhood Ω in X such that K is $\mathcal{O}(\Omega)$ -convex. By the classical theory (see e.g. Chapter 2 in [Hö2]) holomorphic convexity of K is equivalent to the existence of a Stein neighborhood Ω of K and a continuous (or smooth) plurisubharmonic function $\rho \geq 0$ on Ω such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $\Omega \setminus K$. We may take $\Omega = \{\rho < c_1\}$ for some $c_1 > 0$; for any $c \in (0, c_1)$ the sublevel set $\{\rho < c\} \subset\subset \Omega$ is then Stein and Runge in Ω (Sect. 4.3 in [Hö2]).

A \mathcal{C}^1 submanifold M of a complex manifold X is *totally real* in X if for each $p \in M$ the tangent space $T_p M \subset T_p X$ contains no complex line of $T_p X$.

THEOREM 3.1. *Let X be a complex manifold. Assume that $S = K_0 \cup M$ is a closed subset of an open set in X such that K_0 is compact holomorphically convex and $M = S \setminus K_0$ is a \mathcal{C}^1 totally real submanifold. If there exists a*

compact holomorphically convex set $K_1 \subset S$ which is a relative neighborhood of K_0 in S then every compact set K with $K_0 \subset K \subset S$ is holomorphically convex. Fix any such K . Let d be a distance function on X induced by a smooth Riemannian metric. Fix a neighborhood N of K_0 in X . There are a constant $C > 1$ and for any sufficiently small $\epsilon > 0$ a Stein domain $\omega_\epsilon \subset X$ such that

- (i) ω_ϵ contains all points of distance $< \epsilon$ from K ,
- (ii) all points $x \in \omega_\epsilon \setminus N$ have distance $< C\epsilon$ from M .

For $X = \mathbb{C}^n$ this is Theorem 3.1 in [HW] where the result is proved with $C = 2$. In our application (Sect. 4) K_0 is the closure of a strongly pseudoconvex domain and M is attached to K_0 along a submanifold $bM \subset bK_0 =: \Sigma$ which is complex tangential in Σ , i.e., $T_p bM \subset T_p^\perp \Sigma = T_p \Sigma \cap JT_p \Sigma$ for every $p \in bM$, where J is the almost complex structure operator on TX . For such K_0 and M , and with $X = \mathbb{C}^n$, results on holomorphic convexity of $K_0 \cup M$ can be found in [E] (Lemmas 3.3.1. and 3.4.3.), [Ro, Lemma 2], [FK], and other papers.

Proof of Theorem 3.1. By the assumption on K_0 there is a smooth plurisubharmonic function $\rho_0 \geq 0$ in an open Stein neighborhood $U_0 \subset X$ of K_0 such that $\rho_0^{-1}(0) = K_0$. We may assume that the neighborhood N of K_0 in Theorem 3.1 is chosen such that $N \subset U_0$ and $N \cap S \subset K_1$. Choose a sufficiently small $c_2 > 0$ such that $\{\rho_0 < c_2\} \subset\subset N$. Also choose constants $0 < c_0 < c'_0 < c_1 < c'_1 < c_2$.

Since Theorem 3.1 only concerns compact subsets of S , we may assume that S is compact. Since M is totally real, there is a \mathcal{C}^2 strongly plurisubharmonic function $\tau \geq 0$ in an open set $V_0 \supset M$ which vanishes quadratically on M [HW]. Replacing τ by $c\tau$ for a suitable $c > 0$ and shrinking V_0 around M if necessary we may assume that

$$\tau(x) < d(x, M)^2, \quad x \in V_0. \quad (3.1)$$

Since τ vanishes quadratically on M , there is a $C > 0$ such that

$$\tau(x) > d(x, M)^2/C^2, \quad x \in V_0 \cap \{\rho_0 \geq c_0\}. \quad (3.2)$$

Recall that K_1 is assumed to be holomorphically convex. Choose a smooth plurisubharmonic function $\rho_1 \geq 0$ in an open neighborhood U_1 of K_1 such that $\rho_1^{-1}(0) = K_1$ and ρ_1 is strongly plurisubharmonic off K_1 . Since ρ_1 vanishes at least to second order at each point $x \in K_1$, we may assume (after rescaling ρ_1 if necessary) that

$$\rho_1(x) < d(x, S)^2/C^2, \quad x \in U_1 \cap \{\rho \leq c_2\} \quad (3.3)$$

where C is the constant from (3.2). Choose a smooth cut-off function $\chi \geq 0$ on X which equals one on $\{\rho_0 \leq c_0\}$ and satisfies $\text{supp } \chi \subset\subset \{\rho_0 < c'_0\}$. For a

sufficiently small $\delta > 0$ the function $\tau_\delta = \tau - \delta\chi$ is strongly plurisubharmonic on $V_0 \cap \{\rho_0 \geq c_0\}$ (we may need to shrink V_0 around M). Clearly $\tau_\delta = \tau$ on $V_0 \cap \{\rho_0 \geq c'_0\}$. Fix such a δ . Choose an open neighborhood $V \subset U_1 \cup V_0$ of S and let $\rho: V \rightarrow \mathbb{R}_+$ be defined by

$$\rho = \begin{cases} \rho_1 & \text{on } V \cap \{\rho_0 < c_0\}, \\ \max\{\rho_1, \tau_\delta\} & \text{on } V \cap \{c_0 \leq \rho_0 \leq c'_0\}, \\ \tau & \text{on } V \cap \{\rho_0 > c'_0\}. \end{cases}$$

It is easily verified that these choices are compatible provided that the neighborhood V of S is chosen sufficiently small. (When checking the compatibility near $M \cap \{\rho_0 = c'_0\}$ the reader should observe that, by (3.2) and (3.3), we have $\rho_1 < \tau$ and hence $\rho = \tau$ there. Near $M \cap \{\rho_0 = c_0\}$ we have $\rho_1 \geq 0$ while $\tau_\delta < 0$, hence $\rho = \rho_1$.) The function $\rho \geq 0$ is plurisubharmonic, strongly plurisubharmonic on $V \cap \{\rho_0 \geq c'_0\}$ (where it equals τ), and $\rho^{-1}(0) = S$. For every sufficiently small $\epsilon > 0$ the set

$$\omega_\epsilon = \{x \in V: \rho(x) < \epsilon^2\} \subset\subset V$$

is a pseudoconvex open neighborhood of S which satisfies

$$\{x \in X: d(x, S) < \epsilon\} \subset \omega_\epsilon, \quad \omega_\epsilon \setminus N \subset \{x \in X: d(x, M) < C\epsilon\}.$$

The first inclusion follows from $\rho(x) < d(x, S)^2$ which is a consequence of (3.1) and (3.3). The second inclusion is a consequence of (3.2) and the fact that $\rho = \tau$ on $V \setminus N$.

It remains to show that the sets ω_ϵ are Stein. Fix ϵ and choose an increasing convex function $h_\epsilon: (-\infty, \epsilon^2) \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \epsilon^2} h(t) = +\infty$. Then $h_\epsilon \circ \rho$ is a plurisubharmonic exhaustion function on ω_ϵ . In order to obtain a *strongly plurisubharmonic exhaustion function* on ω_ϵ we proceed as follows. Choose a smooth strongly plurisubharmonic function $\xi: U_0 \rightarrow \mathbb{R}$ (such ξ exists since U_0 is Stein). Also choose a smooth cut-off function $\chi \geq 0$ on X such that $\chi = 1$ on $\{\rho_0 \leq c_1\}$ and $\text{supp } \chi \subset\subset \{\rho_0 < c'_1\}$. If $\delta > 0$ is chosen sufficiently small then $\tilde{\rho} = \rho + \delta\chi\xi$ is strongly plurisubharmonic on V . Indeed, on $V \cap \{\rho_0 \leq c_1\}$ we have $\tilde{\rho} = \rho + \delta\xi$ which is strongly plurisubharmonic for every $\delta > 0$; on $V \cap \{\rho_0 > c_1\}$ the function ρ is strongly plurisubharmonic and hence so is $\tilde{\rho}$ provided that δ is chosen sufficiently small. For such δ the function $h_\epsilon \circ \rho + \tilde{\rho}$ is a strongly plurisubharmonic exhaustion on ω_ϵ and hence ω_ϵ is Stein.

This gives a desired Stein neighborhood basis ω_ϵ of S satisfying Theorem 3.1. The same proof applies to any compact subset $K \subset S$ containing K_0 . Alternatively one can apply the above proof with ρ replaced by $\rho + \tau_K$ where $\tau_K \geq 0$ is a smooth function which vanishes to order > 2 on K . This completes the proof of Theorem 3.1.

Remark. Since the function ρ constructed above is plurisubharmonic on the Stein manifold $\omega_{\epsilon_0} = \{\rho < \epsilon_0\}$ for some small $\epsilon_0 > 0$, its sublevel sets $\omega_\epsilon = \{\rho < \epsilon\}$ for $\epsilon \in (0, \epsilon_0)$ are Runge in ω_{ϵ_0} .

THEOREM 3.2. *Let K_0 and $S = K_0 \cup M$ be compact holomorphically convex subsets in a complex manifold X such that $M = S \setminus K_0$ is a totally real m -dimensional submanifold of class \mathcal{C}^r . Assume that $r \geq m/2 + 1$ and let k be an integer satisfying $0 \leq k \leq r - m/2 - 1$. Given an open set $U \subset X$ containing K_0 and a map $f: U \cup M \rightarrow Y$ to a complex manifold Y such that $f|_U$ is holomorphic and $f|_M \in \mathcal{C}^r(M)$, there exist open sets $V_j \subset X$ containing S and holomorphic maps $f_j: V_j \rightarrow Y$ ($j = 1, 2, 3, \dots$) such that, as $j \rightarrow \infty$, the sequence f_j converges to f uniformly on K_0 and in the \mathcal{C}^k -sense on M . If in addition X_0 is a closed complex subvariety of X which does not intersect M and $s \in \mathbb{N}$ then we can choose the approximating sequence such that f_j agrees to order s with f along $X_0 \cap V$ for all $j = 1, 2, 3, \dots$. The analogous result holds for sections $f: X \rightarrow Z$ of any holomorphic submersion $h: Z \rightarrow X$.*

The domains V_j of f_j in Theorem 3.2 may shrink to S as $j \rightarrow \infty$. Although none of the manifolds X, Y, Z in Theorem 3.2 is assumed to be Stein, we shall reduce the proof to that case. In our applications K_0 will be a sublevel set $\{\rho \leq c\}$ of a smooth strongly plurisubharmonic function on X and M will be a smooth totally real handle attached to the hypersurface $bK_0 = \{\rho = c\}$ along a legendrian (complex tangential) sphere.

Proof. When $X = \mathbb{C}^n$, $Y = \mathbb{C}$ (i.e., f is a function) and $k = 0$, Theorem 3.2 coincides with Theorem 4.1 in [HW, p. 11]. For later purposes we recall the sketch of proof which uses the Stein neighborhood basis $\omega_\epsilon \subset X$ of S furnished by Theorem 3.1 in [HW] (compare with Theorem 3.1 above). One first obtains a \mathcal{C}^r extension u of f in a neighborhood of M such that $|D^s(\bar{\partial}u)| = o(d_M^{r-1-s})$, where d_M denotes the distance to M and D^s is the total derivative of order $s \leq r-1$ (Lemma 4.3 in [HW]). Then one solves $\bar{\partial}w_\epsilon = u$ with L^2 estimate on ω_ϵ [Hö1, Hö2]. By the interior elliptic regularity of $\bar{\partial}$ this implies the uniform estimate $|w_\epsilon| = o(\epsilon^{r-m/2-1}) = o(\epsilon^k)$ on $\omega_{c\epsilon}$ for any fixed $c \in (0, 1)$ (the second display on p. 16 in [HW]). If $0 < s \leq k$ we also have $|D^s w_\epsilon| = o(\epsilon^{k-s})$ on $\omega_{c'\epsilon}$ for a fixed $c' \in (0, c)$ (see e.g. Lemma 3.2 and the proof of Proposition 2.3 in [FL]). The function $f_\epsilon = u - w_\epsilon$ is holomorphic on ω_ϵ ; as $\epsilon \rightarrow 0$, f_ϵ converges to f uniformly on K_0 and in the \mathcal{C}^r -sense on M . Since ω_ϵ is Runge in $V := \omega_{\epsilon_0}$ (see the Remark preceding Theorem 3.2), we can approximate f_ϵ by $F_\epsilon: V \rightarrow \mathbb{C}$ and obtain the desired approximating sequence on a fixed open set $V \subset \mathbb{C}^n$. This proves the special case of Theorem 3.2.

For $Y = \mathbb{C}^N$ the result follows immediately by applying it componentwise. The case when X is a Stein manifold (and $Y = \mathbb{C}^N$) reduces to the special case by embedding X as a closed complex submanifold in some \mathbb{C}^n , or by applying the proof in [HW] with the Stein neighborhood basis furnished by Theorem 3.1 above. To prove the general case of Theorem 3.2 we need the following.

LEMMA 3.3. *Let $h: Z \rightarrow X$ be a holomorphic submersion of a complex manifold Z to a complex manifold X . Assume that $S = K_0 \cup M \subset X$ satisfies the hypotheses of Theorem 3.2. Let U be an open set in X containing K_0 and let $f: U \cup M \rightarrow Z$ be a section which is holomorphic on U and smooth of class*

\mathcal{C}^1 on M . Then $f(S)$ has a Stein neighborhood basis in Z .

Proof. We may assume that U is Stein and K_0 is $\mathcal{O}(U)$ -convex. The submanifold $f(M) \subset Z$ is projected by h bijectively onto the totally real submanifold $M \subset X$ and hence $f(M)$ is totally real in Z . Since $f|_U$ is holomorphic, $f(U)$ is a closed complex submanifold of $Z|_U = h^{-1}(U)$ and hence by [De, S] it has an open Stein neighborhood $\tilde{U} \subset Z|_U$. For any compact $\mathcal{O}(U)$ -convex subset $K \subset U$ the set $f(K)$ is holomorphically convex in $f(U)$ and hence (since $f(U)$ is a closed complex submanifold of \tilde{U}) also in \tilde{U} . Applying this with $K = K_0$, and also with $K = (K_0 \cup S) \cap N$ for some compact neighborhood $N \subset U$ of K_0 , we see that $f(S) \subset Z$ satisfies the hypothesis of Theorem 3.1 and hence it has a basis of Stein neighborhoods.

We continue with the proof of Theorem 3.2. Assume that $f: U \cup M \rightarrow Z$ is a section of $h: Z \rightarrow X$ which is holomorphic in U and of class \mathcal{C}^r on M for some $r \geq 1$. Fix a Stein neighborhood $\Omega \subset Z$ of $f(S)$ furnished by Lemma 3.3 and embed Ω as a closed complex submanifold of a Euclidean space \mathbb{C}^N . There is an open neighborhood $\tilde{\Omega} \subset \mathbb{C}^N$ of Ω and a holomorphic retraction $\phi: \tilde{\Omega} \rightarrow \Omega$ [DG]. We consider f as a map into \mathbb{C}^N via the embedding $\Omega \hookrightarrow \mathbb{C}^N$. The special case of Theorem 3.2 gives a sequence of holomorphic maps $g_j: V \rightarrow \mathbb{C}^N$ ($j \in \mathbb{N}$) in an open neighborhood $V \subset X$ of S such that $\lim_{j \rightarrow \infty} g_j|_S = f|_S$ (the convergence is uniform on K_0 and in the \mathcal{C}^k -sense on M). Let $V_j = \{x \in V: g_j(x) \in \tilde{\Omega}\}$; this is a neighborhood of S for sufficiently large $j \in \mathbb{N}$. The sequence of maps $\tilde{f}_j = \phi \circ g_j: V_j \rightarrow \Omega$ then satisfies the conclusion of Theorem 3.2 except that \tilde{f}_j need not be a section of h . This is corrected by projecting the point $\tilde{f}_j(x)$ to the fiber $Z_x = h^{-1}(x)$ by the holomorphic retraction provided by the following

LEMMA 3.4. *Let Ω be a Stein manifold and $h: \Omega \rightarrow V$ a holomorphic submersion onto a complex manifold V . There are an open Stein set $W \subset V \times \Omega$ containing $\Gamma := \{(x, z) \in V \times \Omega: h(z) = x\}$ and a holomorphic retraction $\pi: W \rightarrow \Gamma$ such that $\pi(x, z) = (x, \pi_2(x, z))$ for every $(x, z) \in W$.*

It follows that $h(\pi_2(x, z)) = x$, i.e., $\pi_2(x, \cdot)$ is a holomorphic retraction of an open neighborhood of the fiber $h^{-1}(x)$ in Ω onto $h^{-1}(x)$ for every fixed $x \in V$. Assuming Lemma 3.4 for a moment we set $f_j(x) = \pi_2(x, \tilde{f}_j(x))$ for $j = 1, 2, \dots$; these are holomorphic sections of h in small open neighborhoods $V_j \subset X$ of S satisfying Theorem 3.2.

The version of Theorem 3.2 with interpolation along a subvariety X_0 not intersecting $M = S \setminus K_0$ can be satisfied by using an embedding $\Omega \hookrightarrow \mathbb{C}^N$ and writing $f = f_0 + \sum_{l=1}^{l_0} h_l g_l$ in a neighborhood $V \subset X$ of S , where f_0 and g_l are maps $V \rightarrow \mathbb{C}^N$ such that f_0 is holomorphic in V , $g_l \in \mathcal{C}^r(V)$ is holomorphic over a neighborhood of K_0 (where f is holomorphic), and the functions $h_1, \dots, h_{l_0} \in \mathcal{O}(V)$ vanish to order s on X_0 and satisfy $X_0 \cap V = \{x \in V: h_l(x) = 0, 1 \leq l \leq l_0\}$. (See Lemma 8.1 in [FP2, p. 660] for the details.) Applying Theorem 3.2 with $Y = \mathbb{C}^N$ we approximate each g_l on S by a sequence of sections $g_{l,j}$ ($j = 1, 2, 3, \dots$) holomorphic in a neighborhood

of S in X . This gives a holomorphic sequence $\tilde{f}_j = f_0 + \sum_{l=1}^{l_0} h_l g_{l,j}$ whose restriction to S converges to f as $j \rightarrow \infty$ (in the sense of Theorem 3.2). It remains to compose \tilde{f}_j with the two retractions as above (first by the retraction ϕ onto Ω and then by the retraction furnished by Lemma 3.4) to get a sequence of holomorphic sections $f_j: V_j \rightarrow Z$ ($j = 1, 2, \dots$) which agree with f to order s on X_0 and satisfy $f_j|_S \rightarrow f|_S$ as $j \rightarrow \infty$. This completes the proof of Theorem 3.2 granted that Lemma 3.4 holds.

Proof of Lemma 3.4. Clearly Γ is a closed complex submanifold of $V \times \Omega$. Let $p_1: V \times \Omega \rightarrow V$, $p_2: V \times \Omega \rightarrow \Omega$ denote the respective projections. Observe that $p_2|_{\Gamma}: \Gamma \rightarrow \Omega$ is bijective with the inverse $z \rightarrow (h(z), z)$; hence Γ is Stein and consequently it has an open Stein neighborhood in $V \times \Omega$.

Consider the holomorphic vector subbundle $E \subset T(V \times \Omega)$ whose fiber over (x, z) consists of all vectors $(0, \xi)$, $\xi \in T_z \Omega$. The restricted bundle $E|_{\Gamma}$ can be decomposed as $E|_{\Gamma} = E_0 \oplus N$ where $E_0 = E|_{\Gamma} \cap T\Gamma$ and N is some complementary subbundle. Observe that N is the normal bundle of Γ in $V \times \Omega$ (here we use the hypothesis that $h: \Omega \rightarrow V$ is a submersion). Let N_0 denote the zero section of N . By the Docquier-Grauert theorem [DG] there is a biholomorphic map Φ from an open neighborhood $U \subset N$ of N_0 onto an open neighborhood $W \subset V \times \Omega$ of Γ which maps N_0 onto Γ and maps the fiber $N_{(x,z)} \cap U$ to $\{x\} \times \Omega = p_1^{-1}(x)$ for any $(x, z) \in \Gamma$. (We can obtain such Φ by composing local flows of holomorphic vector fields which are tangent to N .) Choosing U to have convex fibers it follows that Φ conjugates the base projection $U \rightarrow N_0$ to a holomorphic retraction $\pi: W \rightarrow \Gamma$ satisfying the conclusion of Lemma 3.4.

Remark. The loss of smoothness in Theorem 3.2 is due to the method of proof. When $K_0 = \emptyset$ and S is a totally real submanifold of X of class C^r , the optimal C^r -to- C^r approximation was proved by Range and Siu [RS] following the work of many authors. The kernel approach developed in [FLØ] gives precise approximation of smooth diffeomorphisms by biholomorphisms in tubular neighborhoods. Both approaches can likely be adapted to the situation considered here, but we do not need such sharp approximation.

4. Proof of Theorem 2.1.

We first state parts (a) and (b) of Theorem 2.1 in a precise quantitative form; part (c) will be considered later.

THEOREM 4.1. *Let X be a Stein manifold, $K \subset X$ a compact $\mathcal{O}(X)$ -convex subset and Y a complex manifold with $\dim Y \leq \dim X$. Choose a distance function d on Y induced by a complete Riemannian metric. Assume that $(f_0, \iota_0) \in \mathcal{S}(X, Y)$ is such that $f_0|_K: K \rightarrow Y$ is a holomorphic submersion and $\iota_0|_K = Tf_0|_K$. If Y satisfies Property S_n with $n = \dim X$ then for every $\epsilon > 0$ there is a homotopy $(f_t, \iota_t) \in \mathcal{S}(X, Y)$ ($t \in [0, 1]$) from (f_0, ι_0) to some $(f_1, Tf_1) \in \mathcal{S}_{holo}(X, Y)$ such that for every $t \in [0, 1]$, $f_t|_K: K \rightarrow Y$ is a holomorphic submersion, $Tf_t|_K = \iota_t|_K$, and $\sup_{x \in K} d(f_t(x), f_0(x)) < \epsilon$.*

Proof. We shall follow the construction of holomorphic submersions $X \rightarrow \mathbb{C}^q$ in [F1], indicating the necessary changes and additional arguments.

Assume that $U \subset X$ is an open set containing K such that $f_0|_U: U \rightarrow Y$ is a holomorphic submersion and $Tf_0|_U = \iota_0|_U$. Choose a smooth strongly plurisubharmonic Morse exhaustion function ρ on X such that $K \subset \{\rho < 0\} \subset\subset U$ and 0 is a regular value of ρ . We may assume furthermore that in some local holomorphic coordinates near each critical point the function ρ is a quadratic normal form given by (6.1) in [F1].

The construction of the homotopy $(f_t, \iota_t) \in \mathcal{S}(X, Y)$ is done in a countable sequence of *stages*, and each stage consists of finitely many *steps*. We use two different types of steps, one for crossing the noncritical levels of ρ and the other one for crossing a critical level. The selection of sets involved in stages and steps is done in advance and depends only on the exhaustion function ρ . On the other hand, some of the constants in the approximation at each stage (or step) are chosen inductively and depend on the partial solution obtained in the previous steps. The entire construction is quite similar to the proof of the Oka-Grauert principle in [HL2], and especially in [FP1]. We first explain the global scheme; compare with Sect. 6 in [F1].

Let p_1, p_2, p_3, \dots be the critical points of ρ in $\{\rho > 0\} \subset X$, ordered so that $0 < \rho(p_1) < \rho(p_2) < \rho(p_3) < \dots$. Choose a sequence $0 = c_0 < c_1 < c_2 < \dots$ with $\lim_{j \rightarrow \infty} c_j = +\infty$ such that $c_{2j-1} < \rho(p_j) < c_{2j}$ for every $j = 1, 2, \dots$, and the numbers c_{2j-1}, c_{2j} are close to $\rho(p_j)$; the desired closeness will be specified below when crossing the critical level $\{\rho = \rho(p_j)\}$. If there are only finitely many p_j 's, we choose the remainder of the sequence c_j arbitrarily with $\lim_{j \rightarrow \infty} c_j = +\infty$. We subdivide the parameter interval of the homotopy into subintervals $I_j = [t_j, t_{j+1}]$ with $t_j = 1 - 2^{-j}$ ($j = 0, 1, 2, \dots$).

In the j -th stage of the construction we assume inductively that we have a partial solution $(f_t, \iota_t) \in \mathcal{S}(X, Y)$ for $t \in [0, t_j]$ satisfying

$$\sup\{d(f_t(x), f_0(x)): x \in K\} < \epsilon_j, \quad t \in [0, t_j] \quad (4.1)$$

for some $\epsilon_j \leq \epsilon(1 - 2^{-j-1})$, such that $f_{t_j}: \{\rho \leq c_j\} \rightarrow Y$ is a holomorphic submersion and $\iota_{t_j} = Tf_{t_j}$ on $\{\rho \leq c_j\}$. For $j = 0$ these conditions are satisfied with $\epsilon_0 = \epsilon/2$.

Choose a number $\delta_j \in (0, \epsilon_j 2^{-j-1})$ such that any holomorphic map from $\{\rho \leq c_j\}$ to Y which is uniformly δ_j -close to f_{t_j} (measured by the metric d on Y) is a submersion on $\{\rho \leq c_{j-1}\}$. (For $j = 0$ we choose $c_{-1} < 0$ sufficiently close to 0 such that $K \subset \{\rho < c_{-1}\}$.) We also insure that $\delta_j < \delta_{j-1}/2$ where δ_{j-1} is the analogous number from the previous stage; this condition is vacuous for $j = 0$. The goal of the j -th stage is to extend the solution $(f_t, \iota_t) \in \mathcal{S}(X, Y)$ to the interval $t \in I_j = [t_j, t_{j+1}]$ (keeping it unchanged for $t \in [0, t_j]$) such that

$$\sup\{d(f_t(x), f_{t_j}(x)): x \in X, \rho(x) \leq c_j\} < \delta_j/2, \quad t \in I_j, \quad (4.2)$$

$Tf_t = \iota_t$ on $\{\rho \leq c_j\}$ for every $t \in I_j$, and $Tf_{t_{j+1}} = \iota_{t_{j+1}}$ on $\{\rho \leq c_{j+1}\}$. Then $f_{t_{j+1}}$ satisfies the inductive hypothesis on the set $\{\rho \leq c_{j+1}\}$ with $\epsilon_{j+1} = \epsilon_j + \delta_j/2 < \epsilon(1 - 2^{-j-2})$, thus completing the j -th stage.

Assume for a moment that this process can be worked out. For every $t \in [t_j, 1)$ the map f_t is holomorphic on $\{\rho \leq c_j\}$ and $\iota_t = Tf_t$ there. By (4.2) the limits $f_1 = \lim_{t \rightarrow 1} f_t: X \rightarrow Y$ and $\iota_1 = \lim_{t \rightarrow 1} \iota_t = \lim_{t \rightarrow 1} Tf_t: TX \rightarrow TY$ exist uniformly on compacts in X . It follows that $f_1: X \rightarrow Y$ is holomorphic and $\iota_1 = Tf_1$ on X . By the construction we also have

$$d(f_1(x), f_{t_j}(x)) < \delta_j, \quad x \in \{\rho \leq c_j\}, \quad j = 0, 1, 2, \dots$$

The choice of δ_j implies that f_1 is a holomorphic submersion on $\{\rho \leq c_{j-1}\}$. Since this holds for every $j \in \mathbb{N}$, f_1 is a holomorphic submersion of X to Y and hence $(f_1, Tf_1) \in \mathcal{S}_{holo}(X, Y)$. From (4.1) we also get $d(f_1(x), f_0(x)) < \epsilon$ for $x \in K$. This will complete the proof of Theorem 4.1 provided that we prove the inductive stage.

We first consider the *noncritical stages*, i.e., those for which ρ has no critical values on $[c_j, c_{j+1}]$. (In our notation this happens for even values of j .) We solve the problem in finitely many steps of the following kind. We have compact subsets A, B of X satisfying

- (i) $A, B, C := A \cap B$, and $\tilde{A} := A \cup B$ are (the closures of) strongly pseudoconvex domains in X ,
- (ii) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ (the separation property), and
- (iii) there is an open set U in X containing B and a biholomorphic map $\psi: U \rightarrow U'$ onto an open subset $U' \subset \mathbb{C}^n$ containing the cube Q (2.1) such that $\psi(B)$ is a convex subset of Q and $O := \psi(A \cap U) \cap Q$ is a special convex set of the form (2.2).

PROPOSITION 4.2. *Let $A, B \subset X$ satisfy the properties (i)–(iii) above. Assume that $f: A \rightarrow Y$ is a holomorphic submersion. If Y satisfies Property S_n with $n = \dim X$ then for any $\delta > 0$ there is a homotopy of holomorphic submersions $f_t: A \rightarrow Y$ ($t \in [0, 1]$), with $f_0 = f$, such that $\sup_{x \in A} d(f_t(x), f(x)) < \delta$ for every $t \in [0, 1]$ and f_1 extends to a holomorphic submersion $A \cup B \rightarrow Y$.*

For $Y = \mathbb{C}^q$ with $q < \dim X$ this is Lemma 6.3 in [F1]. Its proof also applies in our situation, except that we must replace the use of Proposition 3.3 in [F1] by the assumed Property S_n of Y . We include a sketch of proof.

By hypothesis the map $f \circ \psi^{-1}: U' \rightarrow Y$ is a holomorphic submersion on $O = \psi(A \cap U) \cap Q \subset \mathbb{C}^n$. By Property S_n of Y we can approximate it uniformly on some fixed neighborhood of O as close as desired by a holomorphic submersion $\tilde{g}: Q \rightarrow Y$. (We actually apply S_n with the pair (rO, rQ) of dilated sets for some $r > 1$ close to 1 in order to get approximation on a neighborhood of O .) Since $\psi(B) \subset Q$, the map $g = \tilde{g} \circ \psi: B \rightarrow Y$ is a holomorphic submersion which approximates f uniformly in a neighborhood of $A \cap B$ as close as desired.

By Lemma 5.1 in [F1] we have $f = g \circ \gamma$ for a biholomorphic map γ defined in an open neighborhood of $A \cap B$ in X and uniformly very close to identity map. (Its distance from the identity only depends on the distance of f and g on the given neighborhood of $A \cap B$. The proof of Lemma 5.1 in [F1] holds for arbitrary target manifold Y .) If the approximations are sufficiently close, Theorem 4.1 in [F1] provides a decomposition $\gamma = \beta \circ \alpha^{-1}$ in a neighborhood of $A \cap B$, where α is a biholomorphic map close to the identity in a neighborhood of A in X and β is a biholomorphic map close to the identity in a neighborhood of B . Thus $f \circ \alpha = g \circ \beta$ in a neighborhood of $A \cap B$, and hence the two sides define a holomorphic submersion $\tilde{f}: A \cup B \rightarrow \mathbb{C}^q$ which approximates f on A . Furthermore, there is a homotopy α_t ($t \in [0, 1]$) of biholomorphic maps close to the identity in some fixed neighborhood of A such that α_0 is the identity and $\alpha_1 = \alpha$. (It suffices to embed X as a complex submanifold in some Euclidean space, take the convex linear combinations of α with the identity map, and project this homotopy back to the submanifold X by a holomorphic retraction.) Then $f_t := f \circ \alpha_t: A \rightarrow Y$ is a homotopy of holomorphic submersions from $f_0 = f$ to $f_1 = \tilde{f}$ satisfying Proposition 4.2. This completes the proof.

It remains to explain how Proposition 4.2 is used in the j -th stage of the construction. Since ρ is assumed to have no critical values in $[c_j, c_{j+1}]$, we can obtain the set $\{\rho \leq c_{j+1}\}$ from $\{\rho \leq c_j\}$ by finite number of attachings of ‘convex bumps’ of the type introduced just before Proposition 4.2. We begin with $A_0 = \{\rho \leq c_j\}$ and attach a bump B_0 to get $A_1 = A_0 \cup B_0$; then we attach a new bump B_1 to A_1 to obtain $A_2 = A_1 \cup B_1$, etc., until reaching $A_{k_j} = \{\rho \leq c_{j+1}\}$ (see Lemma 12.3 in [HL1]). The required number of steps k_j depends only on ρ . We also subdivide the parameter interval $I_j = [t_j, t_{j+1}]$ into adjacent subintervals $I_{j,k} = [t_{j,k-1}, t_{j,k}]$ ($k = 1, 2, \dots, k_j$) of equal length, one for every step.

Assume inductively that for some $1 \leq k < k_j$ a solution $(f_t, \iota_t) \in \mathcal{S}(X, Y)$ has already been defined for $t \in [0, t_{j,k-1}]$ such that $f := f_{t_{j,k-1}}$ is a holomorphic submersion from A_{k-1} to Y and $\iota := \iota_{t_{j,k-1}} = Tf$ on this set. Applying Proposition 4.2 on $A = A_{k-1}$ we extend the family of solutions to the next subinterval $t \in I_{j,k}$ such that

$$\sup\{d(f_t(x), f_{t_{j,k-1}}(x): x \in A_{k-1}\} < \delta_j/2k_j$$

(compare with (4.2)). We can also define ι_t for $t \in I_{j,k}$ such that $\iota_t = Tf_t$ on A_{k-1} , $\iota_{t_{j,k}} = Tf_{j,k}$ on A_k , ι_t is homotopic to ι_0 and agrees with ι_0 outside of some small neighborhood of A_k . In k_j steps we extend the family of solutions (f_t, ι_t) to $t \in I_j$ and thus complete the j -th stage.

It remains to consider the *critical stages*, i.e., those for which ρ has a critical point p with $c_j < \rho(p) < c_{j+1}$. For $Y = \mathbb{C}^q$ this is explained in sections 6.2–6.4 in [F1]. Since the proof needs a few modifications, we shall go through it step by step.

Write $c = c_j$. We may assume that c has been chosen as close to $\rho(p)$ as will be needed in the sequel. Near p we use local holomorphic coordinates

on X in which $p = 0$ and ρ is a quadratic normal form (see (6.1) in [F1]). In particular, the stable manifold of $p = 0$ for the gradient flow of ρ is $\mathbb{R}^\nu \subset \mathbb{C}^n$ (the subspace spanned by the real parts of the first ν variables), where ν is the Morse index of ρ at p . To cross the critical level $\rho = \rho(p)$ we perform the following three steps.

Step 1: Extension to a handle. We attach to $\{\rho \leq c\}$ the disc $M \subset \mathbb{R}^\nu \subset \mathbb{C}^n$ (in the local coordinates) such that the attaching sphere $bM \subset \{\rho = c\}$ is complex tangential in the latter hypersurface (Subsect. 6.2 in [F1]). Let (f, ι) be the partial solution obtained after the first $j-1$ stages, so f is a holomorphic submersion from a neighborhood of $\{\rho \leq c\}$ to Y and $\iota = Tf$ there.

LEMMA 4.3. *There is a neighborhood U of $S := \{\rho \leq c\} \cup M$ in X and a smooth map $g: U \rightarrow Y$ which agrees with f in a neighborhood of $K_0 := \{\rho \leq c\}$ such that for every $x \in M$ the differential $dg_x: T_x X \rightarrow T_{g(x)} Y$ is a surjective \mathbb{C} -linear map. Furthermore, (f, ι) can be connected to (g, Tg) by a path (homotopy) in $\mathcal{S}(U, Y)$ which is fixed in a neighborhood of K_0 .*

Proof. The main point is the extension of f and its 1-jet to the handle M such that the above properties are satisfied. One uses Gromov's *convex integration lemma* ([Gr2]; Section 2.4 of [Gr3], especially (D) and (E) in [Gr3, 2.4.1.]; Sect. 18.2 in [EM], especially Corollary 18.2.2.). The details given in [F1] (Lemmas 6.4 and 6.5) for the case $Y = \mathbb{C}^q$ remain valid for arbitrary target manifold Y . The differential relation controlling the problem is *ample in the coordinate directions* and hence Gromov's lemma applies.

It is (only) at this point of the proof that we use the hypothesis on the existence of a fiberwise surjective map $\iota: TX \rightarrow TY$ with base map f . When $q = \dim Y \leq \lfloor \frac{n+1}{2} \rfloor$ it suffices to apply Thom's jet transversality theorem as in [F1], and in this case ι automatically exists.

Step 2: Holomorphic approximation on a handlebody. We denote the result of Step 1 again by (f, ι) ; thus f is a holomorphic submersion from a neighborhood of the strongly pseudoconvex domain $K_0 = \{\rho \leq c\} \subset X$ to Y , it is smooth in a neighborhood of the handle M , $df_x: T_x X \rightarrow T_{f(x)} Y$ is surjective and \mathbb{C} -linear at each point $x \in M$, and $\iota_x = df_x$ for every $x \in S = K_0 \cup M$.

By [F1, Subsect. 6.3] the pair (K_0, M) satisfies the hypothesis of Theorem 3.2 above, and hence there is a holomorphic map \tilde{f} from a neighborhood of S to Y which approximates f uniformly on K_0 and in the \mathcal{C}^1 -sense on M as close as desired. (In [F1] the corresponding approximation result was proved for $Y = \mathbb{C}^q$.) If the approximations are sufficiently close then \tilde{f} is a holomorphic submersion from some open neighborhood of S to Y . We patch the new map with the old one outside of a neighborhood of S .

Step 3: Crossing the critical level of ρ . We denote the result of Step 2 again by f and set $\iota = Tf$ in a neighborhood of $S = \{\rho \leq c\} \cup M$. We patch

ι with ι_0 outside of a neighborhood of S by using a cut-off function in the parameter of the homotopy connecting ι to ι_0 .

By the scheme explained in [F1, Subsect. 6.4] we can approximately extend f across the critical level of ρ at p by performing a noncritical stage with another strongly plurisubharmonic function τ furnished by Lemma 6.7 in [F1]. Afterwards we revert back to the original exhaustion function ρ and proceed to the next noncritical stage associated with ρ . When changing the domain of the solution (first from a neighborhood of the handlebody S to a suitable sublevel set of τ , and later from a higher sublevel set of τ to a supercritical sublevel set ρ) we sacrifice a part of the domain, but the loss is compensated in the next stage. The approximation conditions for the given stage can be satisfied with the appropriate choices of constants at every step. No further changes from [F1] are needed apart from those already explained for a noncritical stage. This completes the proof of Theorem 4.1 and hence of (a), (b) in Theorem 2.1.

To prove part (c) of Theorem 2.1 we perform the same construction for homotopies of submersions connecting a given pair $f_0, f_1: X \rightarrow Y$. Property HS_n is needed to obtain the analogue of Proposition 5.2 for such homotopies. The remaining steps in the proof, including the crossing of a critical level of ρ , require only inessential modifications which we leave to the reader.

Remarks. 1. It is possible to prove a stronger version of Theorem 4.1 with interpolation of a given holomorphic submersion along a closed complex subvariety $X_0 \subset X$; compare with Theorem 2.5 in [F1].

2. Our proof also applies in the equidimensional case $\dim X = \dim Y = n$ as long as Y satisfies Property S_n . Unfortunately we don't know any such example; the main case to be solved is $Y = \mathbb{C}^n$ (see Problems 1–3 in [F1]).

5. Approximation of submersions on subsets of \mathbb{C}^n .

In this section we prove Property S_n and HS_n for certain algebraic manifolds, in particular for those listed in Proposition 2.5.

A complex algebraic subvariety A in an algebraic manifold V will be called *thin* if it does not contain any complex hypersurfaces, i.e., A has complex codimension at least two in V . The following is a key lemma.

LEMMA 5.1. *Assume that $O \subset Q \subset \mathbb{C}^n$ are as in (2.1), (2.2). If A is a thin algebraic subvariety of \mathbb{C}^n with $A \cap O = \emptyset$ then any holomorphic submersion $h: \mathbb{C}^n \setminus A \rightarrow Y$ to a complex manifold Y can be approximated uniformly on O by holomorphic submersions $Q \rightarrow Y$ (and hence by submersions $\mathbb{C}^n \rightarrow Y$).*

Proof. Let $z = (z', z'')$ be the coordinates on \mathbb{C}^n with $z' = (z_1, \dots, z_{n-2})$ and $z'' = (z_{n-1}, z_n)$. After a small linear change of coordinates the projection $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^{n-2}$, $\pi(z', z'') = z'$, is proper when restricted to A . In this situation Lemma 3.4 from [F1] gives for any $\delta > 0$ a holomorphic automorphism ψ of \mathbb{C}^n of the form $\psi(z', z'') = (z', \beta(z', z''))$ satisfying $\sup_{z \in O} |\psi(z) - z| < \delta$ and

$\psi(Q) \cap A = \emptyset$. Then $h \circ \psi: Q \rightarrow Y$ is a holomorphic submersion whose restriction to O is uniformly close to h . This proves Lemma 5.1.

Suppose now that Y is a projective algebraic manifold of dimension q . Given a holomorphic submersion $f: O \rightarrow Y$ from a special convex set $O \subset \mathbb{C}^n$ for some $n > q$, our goal is to approximate f uniformly on O by a rational map $h: \mathbb{C}^n \rightarrow Y$ which is a holomorphic submersion outside of a thin algebraic subvariety $A \subset \mathbb{C}^n$ not intersecting O . If such approximations exist then by Lemma 5.1 we can approximate h (and hence f) uniformly on O by submersions $g: Q \rightarrow Y$, thus proving that Y satisfies Property S_n . In a similar way we establish HS_n .

The possibility of approximating f by h is of course a nontrivial condition on Y which fails for example on Kobayashi-Eisenman hyperbolic manifolds. We shall first establish Property S_n of the projective spaces \mathbb{CP}^q when $n > q$; the proof for Grassmannians and their Zariski open subsets with thin complements will follow the same pattern. (For $Y = \mathbb{C}^q$ see Proposition 3.3 in [F1].)

The quotient projection $\pi: \mathbb{C}_*^{q+1} = \mathbb{C}^{q+1} \setminus \{0\} \rightarrow \mathbb{CP}^q$ is a holomorphic fiber bundle with fiber \mathbb{C}^* , and by adding the zero section we obtain the universal line bundle $L \rightarrow \mathbb{CP}^q$. Assume that $f: O \rightarrow \mathbb{CP}^q$ is a holomorphic map on a compact convex set $O \subset \mathbb{C}^n$. Since O is contractible, the bundle $f^*L \rightarrow O$ is topologically trivial and hence holomorphically trivial [O, G3]. Therefore f^*L admits a nowhere vanishing holomorphic section which can be viewed as a holomorphic map $\tilde{f}: O \rightarrow \mathbb{C}_*^{q+1}$ satisfying $f = \pi \circ \tilde{f}$ (a holomorphic lifting of f). We approximate \tilde{f} uniformly on O by a polynomial map $P: \mathbb{C}^n \rightarrow \mathbb{C}^{q+1}$ and take $h = \pi \circ P: \mathbb{C}^n \setminus P^{-1}(0) \rightarrow \mathbb{CP}^q$; by construction $h|_O$ approximates f .

It remains to show that for a generic choice of P the map h is a submersion outside of a thin subvariety in \mathbb{C}^n . We say that P is *transverse to π* at a point $z \in \mathbb{C}^n$ if $P(z) \neq 0$ and dP_z is transverse to the fiber of π through $P(z)$. Note that $h = \pi \circ P$ is a submersion in a neighborhood of $z \in \mathbb{C}^n$ if and only if P is transverse to π at z . Since $f: O \rightarrow \mathbb{CP}^q$ is a submersion and hence f is transverse to π on O , we may assume (by choosing P sufficiently uniformly close to \tilde{f} on a neighborhood of O) that P is also transverse to π on O . To complete the proof it suffices to show that for a generic choice of P the ‘singularity set’

$$\Sigma_P = P^{-1}(0) \cup \{z \in \mathbb{C}^n \setminus P^{-1}(0): dP_z \text{ is not transverse to } \pi\}$$

is a thin algebraic subset of \mathbb{C}^n provided $n > q$.

Before proceeding we consider also the case when $Y = G_{k,m}$ is the Grassmann manifold consisting of k -dimensional complex subspaces of \mathbb{C}^m . We apply the above proof with the fibration $\pi: V_{k,m} \rightarrow Y$ where $V_{k,m}$ is the Stiefel variety of all k -frames in \mathbb{C}^m acted upon by the group $GL_k(\mathbb{C})$. We can identify $V_{k,m}$ with a Zariski open subset in \mathbb{C}^{km} with thin complement B (which consists of $k \times m$ matrices with rank less than k), and π defines an algebraic foliation \mathcal{F} of \mathbb{C}^{km} which is nonsingular on $V_{k,m}$ and has homogeneous leaves biholomorphic

to $GL_k(\mathbb{C})$. By Grauert's main theorem from [G3] we can lift any holomorphic map $f: O \rightarrow Y$ from a compact convex set $O \subset \mathbb{C}^n$ to a holomorphic map $\tilde{f}: O \rightarrow V_{k,m}$ such that $f = \pi \circ \tilde{f}$ (the argument is essentially the same as above). Next we approximate \tilde{f} by a polynomial map $P: \mathbb{C}^n \rightarrow \mathbb{C}^{km}$. We define the 'singularity set' $\Sigma_P \subset \mathbb{C}^n$ of P as above, except that we replace the origin in the target space by $B = \mathcal{F}_{sing}$. To complete the proof it remains to show that, for $n > \dim Y$, the set Σ_P is thin in \mathbb{C}^n for a generic choice of P . The same proof applies to Zariski open sets Ω with thin complements in $G_{k,m}$; it suffices to add to B the (thin) π -preimage of the complement of Ω .

To complete the proof of Proposition 2.5 (at least the part concerning the Property S_n) we need the following. Fix a positive integer $N \in \mathbb{N}$ and denote by $V = \mathcal{P}(n, m, N)$ the vector space of all polynomial maps $P: \mathbb{C}^n \rightarrow \mathbb{C}^m$ of degree at most N ; note that V may be identified with a Euclidean space.

LEMMA 5.2. *Let \mathcal{F} be an algebraic foliation of codimension q on \mathbb{C}^m with thin singular locus $\mathcal{F}_{sing} \subset \mathbb{C}^m$. Given a polynomial map $P: \mathbb{C}^n \rightarrow \mathbb{C}^m$ let Σ_P consist of all points $z \in \mathbb{C}^n$ such that $P(z) \in \mathcal{F}_{sing}$ or dP_z is not transverse to the leaf of \mathcal{F} through $P(z)$. If $n > q$ then the set Σ_P is thin for all P outside of a proper algebraic subvariety of $\mathcal{P}(n, m, N)$.*

The analogous result holds for any algebraic subsheaf \mathcal{F} (not necessarily integrable) of the tangent sheaf of \mathbb{C}^m for which \mathcal{F}_{sing} (the locus of points where \mathcal{F} is not locally free) is thin. The proof is a standard application of transversality arguments [A, Fo] and is included only for completeness.

Proof of Lemma 5.2. Let $J \simeq \mathbb{C}^{n+m+nm}$ denote the manifold of one-jets of holomorphic maps $\mathbb{C}^n \rightarrow \mathbb{C}^m$. For each holomorphic map $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ the associated one-jet extension is

$$z \in U \rightarrow j_z^1 f = (z, f(z), \partial_1 f(z), \dots, \partial_n f(z)) \in J$$

where $\partial_j = \frac{\partial}{\partial \zeta_j}$. Denote by $p_1: J \rightarrow \mathbb{C}^n$ (resp. $p_2: J \rightarrow \mathbb{C}^m$) the source point (resp. the image point) projection.

Given an algebraic foliation \mathcal{F} on \mathbb{C}^m of codimension q with singularity set \mathcal{F}_{sing} , let $\Sigma_{\mathcal{F}} \subset J$ denote the subset consisting of $p_2^{-1}(\mathcal{F}_{sing})$ together with all one-jets over points $w \in \mathcal{F}_{reg} = \mathbb{C}^m \setminus \mathcal{F}_{sing}$ which are not transverse to $T_w \mathcal{F}$ (the tangent space to the leaf of \mathcal{F} at w). We claim that $\Sigma_{\mathcal{F}}$ is a thin algebraic subset of J provided that \mathcal{F}_{sing} is thin and $n > q$. Clearly $p_2^{-1}(\mathcal{F}_{sing})$ is thin. Furthermore, each point $w \in \mathcal{F}_{reg}$ is contained in a Zariski open set $U \subset \mathbb{C}^n$ such that the restricted foliation $\mathcal{F}|_U$ is defined by one-forms $\omega_1, \dots, \omega_q$ on \mathbb{C}^m with polynomial coefficients which are pointwise independent at w . A one-jet $j_z^1 f$ with $f(z) = w$ fails to be transverse to $T_w \mathcal{F}$ if and only if the $q \times n$ matrix with entries $\langle \omega_j(w), \partial_k f(z) \rangle$ ($1 \leq j \leq q$, $1 \leq k \leq n$) has rank less than q . A simple count shows that the subvariety consisting of all such matrices has codimension $|n - q| + 1$ which is ≥ 2 when $n > q$. This gives over each point

$w \in \mathcal{F}_{reg}$ at least two independent algebraic equations for $\Sigma_{\mathcal{F}}$, thus proving our claim.

We identify $V = \mathcal{P}(n, m, N)$ with the Euclidean space whose elements are collections $c = \{(c_\alpha) : |\alpha| \leq N\}$, where $c_\alpha \in \mathbb{C}^m$ for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$; the correspondence is given by

$$c = \{c_\alpha : |\alpha| \leq N\} \longrightarrow P_c(z) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha \in V.$$

Let $\gamma : \mathbb{C}^n \times V \rightarrow J$ associate to every pair (z, P) the one-jet $j_z^1 P \in J$. Clearly γ is polynomial in z and linear in the coefficients of P . Furthermore, fixing $z \in \mathbb{C}^n$ and the coefficients c_α with $2 \leq |\alpha| \leq N$, γ gives a linear map of maximal rank (a linear submersion) from the space of coefficients $\{c_\alpha : |\alpha| \leq 1\}$ of order ≤ 1 onto the fiber $p_1^{-1}(z) \subset J$. It follows that $\Sigma^1 := \gamma^{-1}(\Sigma_{\mathcal{F}})$ is a thin algebraic subvariety of $\mathbb{C}^n \times V$. Hence for every P outside a proper algebraic subvariety of V the set $\Sigma_P = \{z \in \mathbb{C}^n : (z, P) \in \Sigma^1\}$ is a thin subvariety of \mathbb{C}^n . By construction, Σ_P is the set of point $z \in \mathbb{C}^n$ at which dP_z fails to be transverse to the foliation \mathcal{F} (or $P(z)$ belongs to \mathcal{F}_{sing}). This proves Lemma 5.2.

This establishes Property S_n of any complex manifold Y in Proposition 2.5 for all $n > \dim Y$. The proof that \mathbb{CP}^1 also satisfies Property S_1 requires a slightly different argument as follows. Let $f : O \rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be a holomorphic submersion from a convex set $O \subset \mathbb{C}$ to the Riemann sphere. Then $f^{-1}(\infty) \subset O$ is a finite subset of O which we may assume to be contained in the interior of O . Let $B \subset \mathbb{C}$ be a convex set such that $B \cap f^{-1}(\infty) = \emptyset$, $B \cup O$ is also convex, and $\overline{B \setminus O} \cap \overline{O \setminus B} = \emptyset$. Let $C = B \cap O$. By Theorem 3.1 in [F1] we can approximate $f|_C : C \rightarrow \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$ by entire noncritical functions $g : \mathbb{C} \rightarrow \mathbb{C}$. Proceeding as in the proof of Proposition 4.2 above we find a biholomorphic transition map γ between f and g on a neighborhood of C , decompose $\gamma = \beta \circ \alpha^{-1}$ and thus patch f and g into a submersion $\tilde{f} : B \cup O \rightarrow \mathbb{CP}^1$ which approximates f on O . Furthermore we can arrange that $\tilde{f}^{-1}(\infty) = f^{-1}(\infty)$. In finitely many steps of this kind we approximate f by a submersion $Q \rightarrow \mathbb{CP}^1$ as desired.

We now show that every manifold Y as above also satisfies Property HS_n for all $n > q = \dim Y$. Recall that we have a submersion $\pi : \mathbb{C}^m \setminus B \rightarrow Y$ onto Y where B is thin in \mathbb{C}^m and contains \mathcal{F}_{sing} . Let $O \subset Q \subset \mathbb{C}^n$ be as in (2.1), (2.2). Assume that $f_t : O \subset \mathbb{C}^n \rightarrow Y$ ($t \in [0, 1]$) is a homotopy of holomorphic submersions such that f_0 and f_1 extend to submersions $Q \rightarrow Y$. By the same argument as in the proof of Property S_n we can lift $\{f_t\}$ to a homotopy of maps $\tilde{f}_t : O \rightarrow \mathbb{C}^m$ ($t \in [0, 1]$) which are transverse to the foliation \mathcal{F} defined by π and such that \tilde{f}_0, \tilde{f}_1 are defined and transverse to \mathcal{F} on Q . We approximate $\{\tilde{f}_t\}$ by a homotopy of polynomial maps $P_t : \mathbb{C}^n \rightarrow \mathbb{C}^m$ which also depends polynomially on a parameter $t \in \mathbb{C}$ (the approximation of \tilde{f}_t by P_t takes place on O for $t \in (0, 1)$ and on Q for $t = 0$ and $t = 1$).

Let N be the maximal degree of the maps P_t in the above family, considered as polynomials both in $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Denote by $\tilde{\mathcal{P}}(n, m, N)$ the space of all polynomial maps $(t, z) \in \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^m$ of degree $\leq N$. For every $P \in \tilde{\mathcal{P}}(n, m, N)$ and $t \in \mathbb{C}$ we have $P_t = P(t, \cdot) \in \mathcal{P}(n, m, N)$. The proof of Lemma 5.2 shows that for a generic choice of $P \in \tilde{\mathcal{P}}(n, m, N)$ the ‘singularity set’ $\Sigma_P \subset \mathbb{C}^{1+n}$, consisting of all (t, z) for which the one-jet $j_z^1 P_t$ is not transverse to \mathcal{F} (or $P_t(z) \in B$), is a thin algebraic subvariety of \mathbb{C}^{1+n} . Hence for all but finitely many $t \in \mathbb{C}$ the singularity set $\Sigma_t \subset \mathbb{C}^n$ of the map P_t is also thin. By a small deformation of the segment $[0, 1] \subset \mathbb{R}$ inside \mathbb{C} we may can avoid this finite exceptional set of t ’s, thus obtaining a homotopy $P_t: \mathbb{C}^n \rightarrow \mathbb{C}^m$ of polynomial maps approximating \tilde{f}_t ($t \in [0, 1]$) such that Σ_t is thin for all $t \in [0, 1]$. Hence $h_t = \pi \circ P_t: \mathbb{C}^n \setminus \Sigma_t \rightarrow Y$ is a homotopy of submersions which approximates the original homotopy $f_t: O \rightarrow Y$ uniformly on O , and uniformly with respect to the parameter t .

We can now conclude the proof as in Lemma 5.1: Applying [F1, Lemma 3.4] with the additional parameter $t \in [0, 1]$ we obtain a family of holomorphic automorphisms $\psi_t(z', z'') = (z', \beta_t(z', z''))$ of \mathbb{C}^n , depending smoothly on $t \in [0, 1]$, such that for every t we have $\psi_t(Q) \cap \Sigma_t = \emptyset$, $\psi_t|_O$ is close to the identity map on O , and the maps ψ_0 and ψ_1 are close to the identity map on Q . The homotopy of holomorphic submersions $h_t \circ \psi_t: Q \rightarrow Y$ satisfy all the required properties, except perhaps the interpolation condition at the endpoints $t = 0, 1$ which is easily fixed.

When proving the Property S_n or HS_n for \mathbb{C}^q (or a Zariski open subset with thin complement in \mathbb{C}^q) we proceed as above but skip the first step, i.e, we can directly deal with polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^q$. This completes the proof of Proposition 2.5.

Remark. Our proof of Property HS_n breaks down for multi-parameter families of submersions: we still prove that for a generic choice of the polynomial map $P: \mathbb{C}^k \times \mathbb{C}^n \rightarrow \mathbb{C}^m$ its singularity set $\Sigma_P \subset \mathbb{C}_t^k \times \mathbb{C}_z^n$ (with $t \in \mathbb{C}^k$ being the parameter) is thin; hence for all t outside of a proper algebraic subvariety $Z \subset \mathbb{C}^k$ the singularity set $\Sigma_t \subset \mathbb{C}^n$ of $P_t = P(t, \cdot): \mathbb{C}^n \rightarrow \mathbb{C}^m$ is thin in \mathbb{C}^n . However, when $k > 1$ we may not be able to avoid the exceptional set Z by a small deformation of the parameter cube $[0, 1]^k \subset \mathbb{R}^k$ in \mathbb{C}^k . Such a multi-parameter analogue of Property HS_n would be needed to obtain the complete parametric homotopy principle for submersions $X \rightarrow Y$.

6. A simple proof of the Oka-Grauert-Gromov theorem.

We give a simple proof of the following result from [F3].

THEOREM 6.1. *Let $h: Z \rightarrow X$ be a holomorphic fiber bundle over a Stein manifold X . If the fiber $Y = h^{-1}(x)$ ($x \in X$) admits a finite dominating collection of sprays then the inclusion $\text{Holo}(X, Z) \hookrightarrow \text{Cont}(X, Z)$ of the space of holomorphic sections into the space of continuous sections is a weak homotopy equivalence.*

The spaces of sections are equipped with the compact-open topology. For the definition of ‘dominating families of sprays’ see [F3]. The classical case when Z is a principal holomorphic bundle (with fiber a complex Lie group or homogeneous space) is due to Grauert [G3]. The case when the fiber Y admits a dominating spray is due to Gromov [Gr4, Sec. 2.8], and a detailed proof can be found in [FP1].

Proof. We use the scheme of proof of Theorem 2.1 in Sect. 4 (which is similar to the one in [FP1]). We may assume that the fiber Y is connected.

At a *noncritical step* we have a continuous section $f: X \rightarrow Z$ which is holomorphic in an open neighborhood of a smoothly bounded, compact, strongly pseudoconvex domain $A \subset X$. We attach a small ‘convex bump’ B satisfying

- (i) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ and the union $A \cup B$ is again smoothly bounded strongly pseudoconvex,
- (ii) in some local holomorphic coordinates in a neighborhood $U \subset X$ of B the sets B and $C = A \cap B$ are compact convex subset of \mathbb{C}^n ($n = \dim X$), and
- (iii) the restricted bundle $Z|_U$ is trivial, $Z|_U \simeq U \times Y$.

The set B can be a ‘special convex set’ of the form (2.2) in some local coordinates on X . The goal is to approximate f uniformly on A as close as desired by a continuous section $\tilde{f}: X \rightarrow Z$ which is holomorphic in a neighborhood of $A \cup B$. This is accomplished by the ‘noncritical case’ in [Gr4] or [FP1, Sect. 6] and here we do not propose any changes. We recall the main steps for the sake of the reader.

Let $f_1: U \rightarrow Z|_U \simeq U \times Y$ be any holomorphic section (for instance, a constant section). Since C is convex and hence holomorphically contractible, there is a homotopy $\{f_t\}_{t \in [0,1]}$ of holomorphic sections over a neighborhood $V \subset U$ of C connecting $f_0 := f|_V$ to $f_1|_V$. Since Y admits a finite dominating family of sprays, the homotopy version of the Oka-Weil theorem [F3, Theorem 3.1] gives a uniform approximation of the homotopy $\{f_t\}$ on a smaller neighborhood of C by a homotopy of sections $\{\tilde{f}_t\}_{t \in [0,1]}$ which are holomorphic over a neighborhood of B , with $\tilde{f}_1 = f_1$ and $g := \tilde{f}_0$ very close to f on a neighborhood of C . (This approximation result is essentially due to Grauert [G1, G2]; see also [Gr4] and [FP1].) Applying Theorem 4.1 in [F3] (or Theorem 5.1 in [FP1] when Y admits a dominating spray) we glue the pair of sections f, g into a section \tilde{f} which is holomorphic in an open neighborhood of $A \cup B$ and extends to a continuous section over X . This completes the noncritical step.

A complication in this process arises when crossing a critical level $\{\rho = c_0\}$ of the given strongly plurisubharmonic exhaustion function $\rho: X \rightarrow \mathbb{R}$. In that case we have $A = \{\rho \leq c\}$ for some $c < c_0$ close to c_0 (such that ρ has no critical values on $[c, c_0)$). The set B is a ‘handle’ attached to A such that the attaching set $C = A \cap B$ is no longer contractible. (The union $A \cup B$ is diffeomorphic to the sublevel set $\{\rho \leq c'\}$ for some $c' > c_0$.) In this case we cannot find the desired homotopy $\{f_t\}$ as above. In [FP1] the difficulty

was avoided by applying the noncritical case with an additional parameter to construct $\{f_t\}$, beginning at the ‘core’ of C (which is a totally real torus) and performing ‘approximation and gluing’ until reaching C in finitely many steps (Theorem 4.5 in [FP1]). An alternative method, proposed by Gromov [Gr4] and developed in [FP2], uses a more complicated induction scheme and remains applicable even if the submersion $Z \rightarrow X$ is not locally trivial, as long it admits fiber-dominating sprays over small open subsets of X .

Here we propose a simple alternative way to pass the critical level of ρ by applying Theorem 3.2. Assume that A and f are as above. We attach to A a smooth totally real handle M passing through the critical point p , with $\dim M$ equal the Morse index of p (see Sect. 4, *Step 1: extension to a handle*). By Theorem 3.2 we can approximate f uniformly on $A \cup M$ by a section \tilde{f} which is holomorphic in a neighborhood of $A \cup M$ and continuous on X (compare with *Step 2* in Sect. 4). The proof is completed by *Step 3* in Sect. 4 without any changes (i.e., we use the noncritical case with a different strongly plurisubharmonic function τ in order to pass the critical level of ρ at p , then we revert back to ρ and proceed by the noncritical case to the next critical level of ρ .) All steps adapt easily to the parametric case and hence we obtain the full statement of Theorem 1.4 in [FP1] under the weaker assumption that the fiber admits a dominating family of sprays (instead of a dominating spray). This completes the proof.

Comparing with the proof of Theorem 2.1 we see that the only essential difference lies in the method of local approximation and gluing of pairs of sections. In the proof of Theorem 6.1 we use a dominating family of sprays on the fiber to linearize the problem. On the other hand, in Theorem 2.1 we patch a pair of submersions by decomposing the transition map between them in the source manifold X without using any properties of the target manifold (these are only used for the approximation). The method of passing a critical level is identical in both proofs, except that the construction of submersions requires a maximal rank extension of the one-jet across the handle.

The above may be conceptually the simplest available proof of the Oka-Grauert-Gromov principle, even in Grauert’s classical case concerning principal bundles with homogeneous fibers. Unlike in the earlier papers, holomorphic sections are constructed without using the intermediate technical devices for multi-parameter homotopies of sections.

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